Spectral Dependence of Polarized Radiation due to Spatial Correlations

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Abstract: We study the polarization of light emitted by spatially correlated sources. We show that in general polarization acquires nontrivial spectral dependence due to spatial correlations. The spectral dependence is found to be absent only for a special class of sources where the correlation length scales as the wavelength of light. We further study the cross correlations between two spatially distinct points that are generated due to propagation. It is found that such cross correlation leads to sufficiently strong spectral dependence of polarization which can be measured experimentally.

In a series of interesting papers Wolf [1–3] showed that in general the spectrum of electromagnetic radiation does not remain invariant under propagation, even through vacuum. The effect arises if the source has spatial correlations. The phenomenon was later confirmed experimentally [4–6] and has been a subject of considerable interest [7,8]. Further investigations of the source correlation effects have been done in the time domain theoretically [9] and experimentally [10]. Several applications of the effect have also been proposed [11–15]. In a related development it has been pointed out that spectral changes also arise due to static scattering [16–20] and dynamic scattering [21–25].

In the current paper we study the spectral dependence of polarized radiation that can arise due to spatial correlations. We are interested in sources where the emission from different points in the source are correlated i.e. the phase and amplitude shows systematic dependence on the position at the source. This subject has attracted considerable attention recently [26–32]. In the present paper we are interested in obtaining the form of the correlation matrix for which the polarization in the far zone will not show any spectral dependence. One physical example we have in mind is radiation from a plasma in the presence of background magnetic field. The motion of charged particles at different spatial locations is in general correlated and will lead to spatially correlated radiation. In a separate paper [33] we have also studied the angular dependence of the polarization in the far zone due to spatial correlations.

Consider a spatially extended 3-D source of polarized radiation, as shown in figure 1, characterized by the charge density $\rho^{(r)}(\mathbf{r},t)$ and current density $\mathbf{J}^{(r)}(\mathbf{r},t)$. The electric and magnetic field vectors, $\mathbf{E}^{(r)}$ and $\mathbf{B}^{(r)}$, generated by this source, can be written as [34],

$$\mathbf{E}^{(r)}(\mathbf{R},t) = \frac{1}{4\pi\epsilon_0}\int d^3r\Bigg(\frac{\mathbf{S}}{S^3}\left[\rho^{(r)}(\mathbf{r},t')\right]_{\rm ret} + \frac{\mathbf{S}}{cS^2}\left[\frac{\partial\rho^{(r)}(\mathbf{r},t')}{\partial t'}\right]_{\rm ret} - \frac{1}{c^2S}\left[\frac{\partial \mathbf{J}^{(r)}(\mathbf{r},t')}{\partial t'}\right]_{\rm ret}\Bigg) \quad (1)$$

$$\mathbf{B}^{(r)}(\mathbf{R},t) = \frac{\mu_0}{4\pi}\int d^3r\Bigg(\left[\mathbf{J}^{(r)}(\mathbf{r},t')\right]_{\rm ret}\times\frac{\mathbf{S}}{S^3} + \left[\frac{\partial \mathbf{J}^{(r)}(\mathbf{r},t')}{\partial t'}\right]_{\rm ret}\times\frac{\mathbf{S}}{cS^2}\Bigg) \quad (2)$$

where $\mathbf{S} = \mathbf{R}-\mathbf{r}$, $S = |\mathbf{S}|$, $[f(\mathbf{r},t')]_{\rm ret} = f(\mathbf{r}, t - S/c)$, c = speed of light and $\mathbf{R}$ and $\mathbf{r}$ are the position vectors of the observation point Q (Fig 1) and a particular point P on the source respectively. Let $\mathbf{E}(\mathbf{R},\omega)$, $\mathbf{B}(\mathbf{R},\omega)$, $\rho(\mathbf{r},\omega)$ and $\mathbf{J}(\mathbf{r},\omega)$ be the analytic signals associated with the Fourier transforms of the electric field, magnetic field, charge and current densities respectively. In the radiation zone, $\mathbf{E}(\mathbf{R},\omega)$ and $\mathbf{B}(\mathbf{R},\omega)$ are given by,

$$\mathbf{E}(\mathbf{R},\omega) = \frac{i\omega}{4\pi\epsilon_0 c^2}\int d^3r\frac{e^{i\omega S/c}}{R}\left[\mathbf{J}(\mathbf{r},\omega) - \mathbf{J}(\mathbf{r},\omega)\cdot\hat{\mathbf{R}}\hat{\mathbf{R}}\right] \quad (3)$$

$$\mathbf{B}(\mathbf{R},\omega) = \frac{-i\omega\mu_0}{4\pi c}\int d^3r\frac{e^{i\omega S/c}}{R}\mathbf{J}(\mathbf{r},\omega)\times\hat{\mathbf{R}} \quad (4)$$

where $\hat{\mathbf{R}} = \mathbf{R}/R$ and $R = |\mathbf{R}|$. In obtaining this result we have used the current conservation equation and kept only the leading order terms in $1/R$. The dimensions of the source (Fig. 1) are assumed to be much smaller than its distance to the observation point Q and hence dropping higher orders in $1/R$ is reasonable. We see that the electric and magnetic fields are orthogonal to one another as well as to the propagation vector $\mathbf{k} = \hat{\mathbf{R}}\omega/c$. Therefore at the far away point Q only the θ,ϕ components of the electric and magnetic field vector are nonzero.

We are interested in evaluating the coherency matrix $J_{ij}(\mathbf{R},\omega)$ at the point Q in the radiation zone. We can relate this in terms of the source correlation function

$$W^S_{ij}(\mathbf{r},\mathbf{r}';\omega)\delta(\omega-\omega') =< J_i^*(\mathbf{r},\omega)J_j(\mathbf{r}',\omega') >, \quad (5)$$

where the angular brackets represent ensemble averages. Here we have made the standard assumption that the source fluctuations are represented by a stationary statistical ensemble, atleast in the wide sense [8]. The correlation function of the electric field W_{ij} is then also given by,

$$W_{ij}(\mathbf{R},\mathbf{R}';\omega)\delta(\omega-\omega') =< E_i^*(\mathbf{R},\omega)E_j(\mathbf{R}',\omega') > \quad (6)$$

The coherency matrix $J_{ij}(\mathbf{R},\omega)$ is given by,

$$J_{ij}(\mathbf{R},\omega) = W_{ij}(\mathbf{R},\mathbf{R},\omega)\ . \quad (7)$$

A straightforward calculation using Eqs. 3 and 4 gives,

$$J_{ij}(\mathbf{R},\omega) = Z(\omega)\int d^3r d^3r' \frac{e^{ik(S-S')}}{SS'} \xi_{il} W^S_{lm}(\mathbf{r},\mathbf{r}',\omega)\xi_{mj} \tag{8}$$

where,

$$\xi_{ij} = -\hat{R}_i\hat{R}_j + \delta_{ij} \ , \tag{9}$$

$k = \omega/c$, $S' = |\mathbf{R} - \mathbf{r}'|$ and $Z(\omega)$ is an overall normalization factor which will not play any role in our analysis. In Eq. 8 as well in the rest of the paper summation over repeated indices is understood unless otherwise stated. We point out that since only the transverse (θ, ϕ) components of the electric field vector are nonzero, the subscripts on J as well as W^S refer only to these components. The $(S - S')$ term in the exponent can be replaced by $\hat{\mathbf{R}}\cdot\boldsymbol{\Delta}$, where $\boldsymbol{\Delta} = \mathbf{r}' - \mathbf{r}$.

The matrix $W^S_{ij}(\mathbf{r},\mathbf{r}',\omega)$ measures the correlations between two spatially distinct points. We are interested in studying the polarization at a point Q in the far zone when the cross correlation matrix has nontrivial dependence on $\mathbf{r}$ and $\mathbf{r}'$. It is convenient to express the matrix W^S_{ij} in terms of the variables $\mathbf{r_a} = (\mathbf{r}+\mathbf{r}')/2$ and $\boldsymbol{\Delta}$ instead of $\mathbf{r}$ and $\mathbf{r}'$. We rewrite Eq. 8 in terms of $\mathbf{r_a}$ and $\boldsymbol{\Delta}$,

$$J_{ij}(\mathbf{R},\omega) = Z(\omega)\int d^3r_a d^3\Delta \frac{e^{ik\hat{\mathbf{R}}\cdot\boldsymbol{\Delta}}}{R^2} \xi_{il} W^S_{lm}(\mathbf{r_a},\boldsymbol{\Delta},\omega)\xi_{mj} \ . \tag{10}$$

The limits of integration over the variables $\mathbf{r_a}$ and $\boldsymbol{\Delta}$ are assumed to extend over all space. In obtaining Eq. 10 we have ignored higher order terms in $|\boldsymbol{\Delta}|/R$ in the exponent which is reasonable if the observation point is far enough away.

In this paper we are primarily interested in studying the $\boldsymbol{\Delta}$ dependence of the correlation matrix and hence consider only sources for which the $\mathbf{r_a}$ dependence factorizes, that is,

$$W^S_{ij}(\mathbf{r_a},\boldsymbol{\Delta},\omega) = j_{il}(\mathbf{r_a})G_{lj}(\boldsymbol{\Delta},\omega) \ . \tag{11}$$

This equation defines the two matrices, $j_{ij}(\mathbf{r_a})$ and $G_{ij}(\boldsymbol{\Delta},\omega)$. Physically this factorization means that the spatial correlations as well as the spectral response of different regions of the source are identical to one another. However the polarization of light emitted by different points on the source need not be identical. Furthermore in order to focus on the spectral dependence of polarization which arises due to spatial correlations we assume that,

$$G_{ij}(\boldsymbol{\Delta} = 0,\omega) = N\mathcal{S}(\omega)\delta_{ij} \tag{12}$$

where N is a normalization factor, $\mathcal{S}(\omega)$ is the spectrum of light emitted by any point on the source and δ_{ij} is the standard Kronecker delta. Physically Eq. 12 means that if we ignore spatial correlations, the polarization of light emitted by the source has no spectral dependence. If the spatially distinct points are independent then,

$$G_{ij}(\boldsymbol{\Delta},\omega) = \delta^3(\boldsymbol{\Delta})\mathcal{S}(\omega)\delta_{ij} \ . \tag{13}$$

In this case we find that the resultant matrix J is given by

$$J_{ij}(\mathbf{R},\omega) = \frac{\mathcal{S}(\omega)}{R^2}\int d^3r_a \xi_{il} j_{lm}(\mathbf{r_a})\xi_{mj} \tag{14}$$

i.e. an incoherent integral over the entire source. In this case we find, as expected, that the resulting polarization at Q has no spectral dependence.

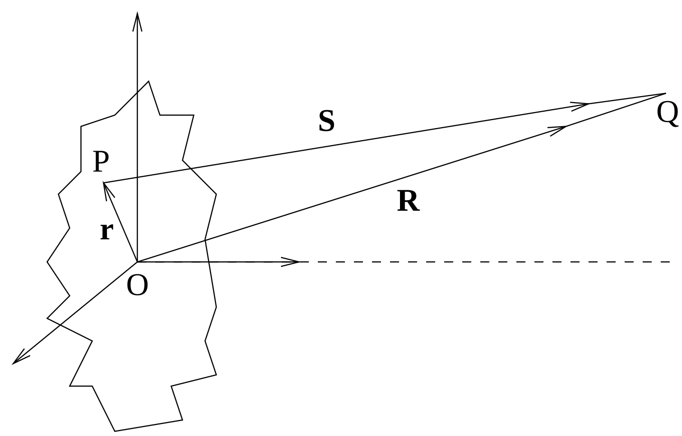

FIG. 1. Schematic illustration of spatially extended 3-D source of polarized radiation and the observation point Q located at position vector $\mathbf{R}$ from the source. P represents any point on the source located at position vector $\mathbf{r}$ with respect to the origin O of the coordinate system and at a distance $S = |\mathbf{S}|$ from Q.

In general, however, the polarization observed at the far away point Q does acquire spectral dependence purely due to spatial correlations. We next obtain conditions on the correlation matrix W^S under which this spectral polarization dependence is absent. We consider only those sources which satisfy the conditions given in Eq. 11 and 12 since we are interested in isolating the spectral dependence that arises only due to spatial correlations.

If the coherency matrix factorizes into a function of ω times a matrix independent of ω, that is,

$$J_{ij}(\mathbf{R},\omega) = M_{ij}(\mathbf{R})h(\omega) \ , \tag{15}$$

where M_{ij} is a matrix independent of ω and $h(\omega)$ is a function of ω, then the polarization observed at Q will have no spectral dependence. In order to obtain the functional form of $W^S_{lm}(\mathbf{r_a},\boldsymbol{\Delta},\omega)$ which can lead to coherency matrix of the form given in Eq. 15 we substitute Eq. 11 into Eq. 10. We can then express J_{ij} as

$$J_{ij}(\mathbf{R},\omega) = Z(\omega)\xi_{il}\frac{J^0_{ln}\tilde{G}_{nm}(\mathbf{k},\omega)}{R^2}\xi_{mj} \tag{16}$$

where

$$J^0_{ln} = \int d^3r_a j_{ln}(\mathbf{r_a}) \tag{17}$$

and

$$\tilde{G}_{nm}(\mathbf{k},\omega) = \int d^3\Delta G_{nm}(\mathbf{\Delta},\omega)e^{i\mathbf{k}\cdot\Delta}, \tag{18}$$

with $\mathbf{k} = k\hat{\mathbf{R}}$. Hence we see that $\tilde{G}_{nm}(\mathbf{k},\omega)$ is the Fourier transform of $G_{nm}(\mathbf{\Delta},\omega)$. We can therefore also write

$$G_{nm}(\mathbf{\Delta},\omega) = \int \frac{d^3k}{(2\pi)^3}\tilde{G}_{nm}(\mathbf{k},\omega)e^{-i\mathbf{k}\cdot\mathbf{\Delta}} \tag{19}$$

We point out that the integration in the above equation is performed treating ω to be independent of $\mathbf{k}$. In order that the polarization at the point Q has no spectral dependence, $\tilde{G}_{nm}(\mathbf{k},\omega)$ has to be of the form

$$\tilde{G}_{nm}(\mathbf{k},\omega) = \tilde{A}_{nm}(\mathbf{k}/\omega)h(\omega) \tag{20}$$

or

$$\tilde{G}_{nm}(\mathbf{k},\omega) = \delta_{nm}\tilde{H}(\mathbf{k},\omega)\ , \tag{21}$$

where the matrix $\tilde{A}_{nm}$ depends on $\mathbf{k}$ and ω only through the combination $\mathbf{k}/\omega$ and $H(\mathbf{k},\omega)$ is some function of $\mathbf{k},\omega$. Substituting Eq. 20 into Eq. 19 we find that,

$$G_{lm}(\mathbf{\Delta},\omega) = h(\omega)G_{lm}(\omega\mathbf{\Delta})\ \ , \tag{22}$$

which is analogous to the scaling law obtained by Wolf [1] in his analysis of spectral shifts from spatially correlated sources. Alternatively substituting the factorized form Eq. 21 into Eq. 19 we find that,

$$G_{lm}(\mathbf{\Delta},\omega) = \delta_{lm}H(\mathbf{\Delta},\omega)\ , \tag{23}$$

where $H(\mathbf{\Delta},\omega)$ is the Fourier transform of $\tilde{H}(\mathbf{k},\omega)$. We therefore find that in order that the polarization in the radiation zone does not acquire spectral dependence, the correlation matrix $G_{lm}(\mathbf{\Delta},\omega)$ has to be of the form given in Eq. 22 or Eq. 23.

We next consider a specific example and calculate the spectral dependence arising due to correlations. We consider a planar circular source of radius a, which is spatially uncorrelated. The source emits polarized radiation such that its coherency matrix is given by,

$$J(\rho,\phi) = A\begin{pmatrix} \sin^2\phi & -\sin\phi\cos\phi \\ -\sin\phi\cos\phi & \cos^2\phi \end{pmatrix} \tag{24}$$

where A is a constant and ρ,ϕ are the polar coordinates of any point at the position vector $\mathbf{r}$ on the source. The source luminosity is independent of position and the polarization vectors point along $\hat{\phi}$. We point out that the source has been constructed such that at any point close to the axis of symmetry of the source the integrated polarization is zero.

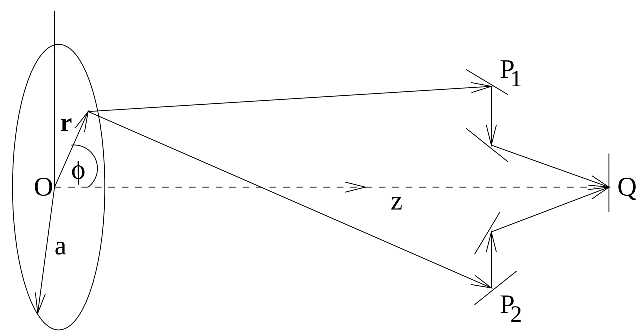

FIG. 2. Schematic illustration of a 2-D spatially extended source of polarized light. $P_1(\mathbf{R_1})$ and $P_2(\mathbf{R_2})$ are any points close to the axis of the source. After reflection from points P_1 and P_2, light reaches the observation point Q, where its spectrum is measured. Due to spatial correlation between P_1 and P_2, the polarization of light at Q displays a nontrivial spectral dependence.

As is well known, although the source is uncorrelated, the cross correlation between any two points P_1 and P_2 need not be zero due to the Van Cittert-Zernike theorem [8]. We consider the experimental arrangement shown in Fig. 2, where the light after being reflected from P_1 and P_2 is observed at the point Q. We are interested in the spectral dependence of the polarization observed at Q. The cross correlation matrix between any two points $P_1(\mathbf{R_1})$ and $P_2(\mathbf{R_2})$ in the far zone close to symmetry axis of the source, is given by,

$$W_{ij}(\mathbf{R_1},\mathbf{R_2},\omega) = \left(\frac{k}{2\pi}\right)^2\int d^2r J_{ij}(\mathbf{r})\frac{e^{-ik(\rho/R)L\cos(\phi-\psi)}}{R^2} \tag{25}$$

where $L\cos\psi = x_2 - x_1$, $L\sin\psi = y_2 - y_1$, (x_1, y_1) and (x_2, y_2) are the cartesian coordinates of the projections of $\mathbf{R_1}$ and $\mathbf{R_2}$ respectively on the plane of the source and $R = R_1 = R_2$ is the distance of the point O on the source from points P_1 and P_2 which have been assumed to be placed symmetrically for simplicity. In obtaining Eq. 25 we have followed the treatment given in Ref. [8] for the calculation of cross correlation between two points P_1 and P_2 close to the symmetry axis of a spatially uncorrelated source. Lack of spatial correlation implies that the cross correlation matrix at the source $W^S_{ij}(\mathbf{r_1},\mathbf{r_2},\omega) = J_{ij}(\mathbf{r_1},\omega)\delta^2(\mathbf{r_2}-\mathbf{r_1})$. We have further assumed that $J_{ij}(\mathbf{r_1},\omega)$ has no spectral dependence. The integral in Eq. 25 is over the source and we are using polar coordinates $x = \rho\cos\phi$ and $y = \rho\sin\phi$.

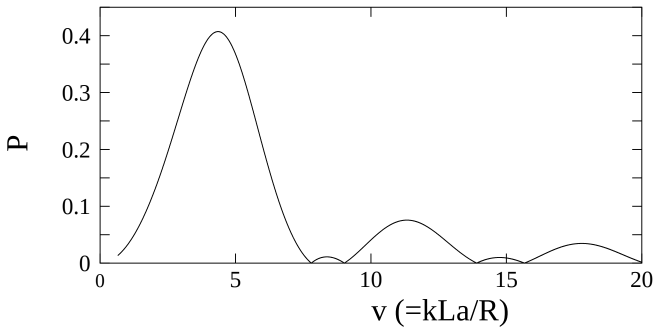

FIG. 3. The degree of polarization P at the observation point Q due to the source model specified by Eq. 24 as a function of $v = kLa/R$. Here k is the wavenumber of light, L is the distance between points P_1 and P_2 (see Fig. 2), a is the radius of the primary 2-D source and R is the distance of the points P_1 and P_2, assumed to be located symmetrically, from the center of the source. The 2-D source is spatially uncorrelated with uniform intensity and polarization vectors pointing along $\hat{\phi}$ at any point (ρ, ϕ) on the source.

Following the treatment given in [8] we can calculate the cross correlation matrix and hence the Stokes parameters at the point of observation. For the source under consideration the result can be obtained analytically. We find, upto an overall common factor $Aa^2k^2/2\pi R^2$

$$s_0 = 1 + 2J_1(v)/v \ ,$$

$$s_1 = \frac{2}{v^2}\frac{(y_2-y_1)^2-(x_2-x_1)^2}{L^2}[vJ_1(v) - 2(1-J_0(v)] \ ,$$

$$s_2 = -\frac{4}{v^2}\frac{(x_2-x_1)(y_2-y_1)}{L^2}[vJ_1(v) - 2(1-J_0(v)] \ ,$$

$$s_3 = 0$$

where $v = kLa/R$. We therefore find that the wave is linearly polarized at the point of observation Q with the degree of polarization given by,

$$P = 2\frac{|-vJ_1(v) + 2 - 2J_0(v)|}{v^2 + 2vJ_1(v)} \qquad (26)$$

which has nontrivial spectral dependence. The orientation of the linear polarization vector, given by,

$$\tan(2\psi) = \frac{2(x_2-x_1)(y_2-y_1)}{(x_2-x_1)^2-(y_2-y_1)^2} \ , \qquad (27)$$

no spectral dependence. The calculated degree of polarization for this example is plotted in Fig. 3. We clearly see that it is a very significant effect and can be observed experimentally. The orientation of the linearly polarized component depends on the positions of P_1 and P_2 and contains information about the polarization profile of the source.

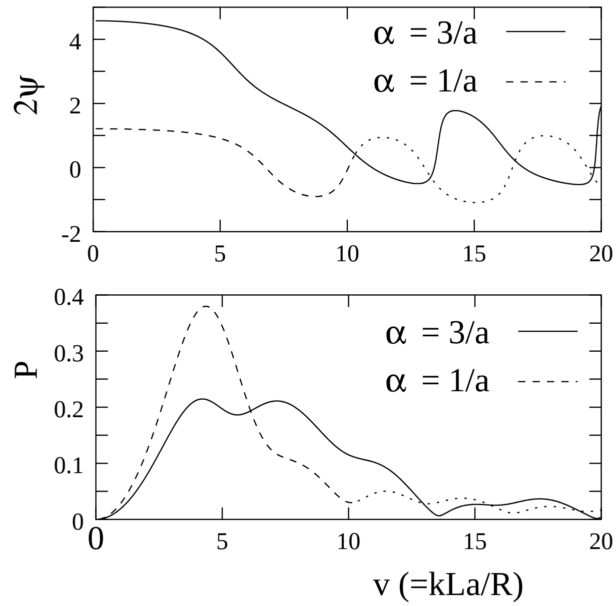

FIG. 4. (a) The linear polarization angle ψ and (b) the degree of polarization P at the observation point Q as a function of $v = kLa/R$. Here k is the wavenumber of light, L is the distance between points P_1 and P_2 (see Fig. 2), a is the radius of the primary 2-D source and R is the distance of the points P_1 and P_2, assumed to be located symmetrically, from the center of the source. The primary 2-D source is spatially uncorrelated with uniform intensity, with its polarization specified by Eq. 24 with ϕ replaced by $\phi + \alpha\rho$, where α is a parameter. The polarization vectors make an angle $\alpha\rho$ with $\hat{\phi}$ at any point (ρ, ϕ) on the source. We study two representative choices of the parameter $\alpha = 1/a, 3/a$.

We next study a somewhat more complicated source for which the coherency matrix at any point $\mathbf{r} = (\rho, \phi)$ is same as that given in Eq. 24, with ϕ replaced by $\phi + \alpha\rho$, where α is a parameter. In this case we numerically calculate the Stokes parameter. The spectral dependence of the degree of polarization and the orientation of the linear polarization is shown in Fig. 4 for some representative choices of the parameter α. The plot uses $x_2 - x_1 = 1.0$ and $y_2 - y_1 = -0.2$ in arbitrary units. The state of polarization in the far zone only depends on the dimensionless quantities $v = kLa/R$, $(x_2 - x_1)/L$, $(y_2 - y_1)/L$ and αa where $L^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$. We find that in this case both the orientation angle of the linear polarization vector and the degree of polarization show a dramatic spectral variation. The Stokes parameter s_3 vanishes in this case also showing that there is no circularly polarized component at the point Q.

The effect discussed in this paper is observable experimentally by using a primary source which has spatial correlations or by generating the correlations due to propagation as shown in the above example. We assume an experimental arrangement shown in Fig. 2. We first measure the spectral dependence of polarization at the point Q due to the waves emerging from the secondary sources

P_1 and P_2 individually. The effect of spatial correlations can then be determined by measuring the polarization at Q due to interference of the waves emerging from P_1 and P_2.

We conclude that spatially correlated sources of polarized radiation generically display nontrivial spectral dependence of the state of polarization in the far zone. This dependence goes away if the correlation matrix displays a scaling law or factorizes into a constant matrix and a function of the relative coordinate and frequency.

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